For a matter-dominated model \( q_0 = \Omega_0/2 \); for a radiation-dominated model \( q_0 = \Omega_0 \); and for a vacuum-dominated model \( q_0 = -\Omega_0 \) (in a vacuum-dominated model the expansion is accelerating, \( \dot{R} > 0 \)).

### 3.2 The Expansion Age of the Universe

The Friedmann equation may be integrated to give the age of the Universe in terms of present cosmological parameters. The energy density scales as \( \rho/\rho_0 = (R/R_0)^{-3} \) for a matter-dominated (MD) Universe, and scales as \( \rho/\rho_0 = (R/R_0)^{-4} \) for a radiation-dominated (RD) Universe. The Friedmann equation becomes

\[
\left( \frac{\dot{R}}{R_0} \right)^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3} \rho_0 \frac{R_0}{R} \quad \text{(MD)}
\]

\[
\left( \frac{\dot{R}}{R_0} \right)^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3} \rho_0 \left( \frac{R_0}{R} \right)^2 \quad \text{(RD).} \quad (3.20)
\]

Using the fact that \( k/R_0^2 = H_0^2(\Omega - 1) \), the age of the Universe as a function of \( R_0/R = 1 + z \) is given by

\[
t = \int_0^{R(t)} \frac{dR'}{R'}
\]

\[
= H_0^{-1} \int_0^{(1+z)^{-1}} \frac{dx}{[1 - \Omega_0 + \Omega_0 x^{-1}]^{1/2}} \quad \text{(MD)}
\]

\[
= H_0^{-1} \int_0^{(1+z)^{-1}} \frac{dx}{[1 - \Omega_0 + \Omega_0 x^{-2}]^{1/2}} \quad \text{(RD).} \quad (3.21)
\]

As expected, the time scale for the age of the Universe is set by the Hubble time \( H_0^{-1} \).

As noted above, it is conventional to define the zero of time to be that time when the scale factor \( R \) extrapolates to zero. Of course, this is arbitrary as all the equations governing the Universe are invariant under time translation. As we will see, in the calculation of physical quantities, the timescale enters as \( H^{-1} \), which is independent of the zero of time. In calculating the age of the Universe, a potential problem arises: Earlier than some time, say \( t_\tau \) (more accurately, for \( R \) less than some \( R_\tau \)), our knowledge of the Universe is uncertain, and so the time elapsed from \( R = 0 \)
to $R = R_\tau$ cannot be reliably calculated. However, the contribution to the age of the Universe from $R = 0$ to $R = R_\tau$ is very small, providing $R \sim t^n \ (n < 1)$ during this period: $\int_0^{R_\tau} dR/\dot{R} \sim H^{-1}$. Put another way, so long as $R \sim t^n \ (n < 1)$, most of the time elapsed since $R = 0$ accumulated during the most recent few Hubble times. Unless there was some very long early epoch where $R \sim t^n \ (n > 1)$, the “error” in calculating the age of the Universe at late times is negligible, and of course the precise age of the Universe is irrelevant for any microphysical calculation since the only timescale that enters such calculations is the Hubble time at the epoch of interest.

The integrals in (3.21) are easily evaluated. Consider the matter-dominated case first. In terms of $\Omega_0$ and $z$,

$$
t = H_0^{-1} \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \times \left[ \cos^{-1} \left( \frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0} \right) - \frac{2(\Omega_0 - 1)^{1/2}(\Omega_0 z + 1)^{1/2}}{\Omega_0 (1 + z)} \right] \tag{3.22}
$$

for $\Omega_0 > 1$, and

$$
t = H_0^{-1} \frac{\Omega_0}{2(1 - \Omega_0)^{3/2}} \times \left[ - \cosh^{-1} \left( \frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0} \right) + \frac{2(1 - \Omega_0)^{1/2}(\Omega_0 z + 1)^{1/2}}{\Omega_0 (1 + z)} \right] \tag{3.23}
$$

for $\Omega_0 < 1$. For $\Omega_0 = 1$, $t = (2/3)H_0^{-1}(1 + z)^{-3/2}$. The present age of a matter dominated Universe, $t_0$, is given by the above expressions with $z = 0$

$$
t_0 = H_0^{-1} \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \left[ \cos^{-1}(2\Omega_0^{-1} - 1) - \frac{2}{\Omega_0} (\Omega_0 - 1)^{1/2} \right] \tag{3.24}
$$

for $\Omega_0 > 1$, and

$$
t_0 = H_0^{-1} \frac{\Omega_0}{2(1 - \Omega_0)^{3/2}} \left[ \frac{2}{\Omega_0} (1 - \Omega_0)^{1/2} - \cosh^{-1}(2\Omega_0^{-1} - 1) \right] \tag{3.25}
$$

As an example, consider a Universe that “began” in a vacuum-dominated phase; then early on $R \sim \exp(\lambda t)$, and the time back to the singularity is infinite.
for $\Omega_0 < 1$. If $\Omega_0 = 1$, $t_0 = (2/3)H_0^{-1}$.

The age of the Universe is a decreasing function of $\Omega_0$: Larger $\Omega_0$ implies faster deceleration, which in turn corresponds to a more rapidly expanding Universe early on. In the limit $\Omega_0 \to 0$, $t \to H_0^{-1}(1 + z)^{-1} = 9.78 \times 10^9 h^{-1}(1+z)^{-1}$ years. Expressions (3.24) and (3.25) can be expanded about $\Omega_0 = 1$:

$$t_0 \simeq \frac{2}{3}H_0^{-1} \left[ 1 - \frac{1}{5}(\Omega_0 - 1) + \cdots \right]. \tag{3.26}$$

At early times, $(1 + z) \gg \Omega_0^{-1}$, both (3.22) and (3.23) reduce to

$$t \simeq \frac{2}{3}(1 + z)^{-3/2}H_0^{-1}\Omega_0^{-1/2}. \tag{3.27}$$

For large $1+z$ the dependence of $t$ upon $\Omega_0$ simplifies greatly; this occurs because for $(1 + z) \gg \Omega_0^{-1}$ the $k/R^2$ term becomes negligible compared to the matter density term, and $t \to (2/3)H^{-1} \sim \rho^{-1/2} \propto z^{-3/2}H_0^{-1}\Omega_0^{-1/2}$.

The present age of a matter-dominated, $\Omega_0 = 1$ Universe is $6.52 \times 10^9 h^{-1}$ years. If $h$ is not too much larger than $1/2$, this age is consistent
3.2 Expansion Age

with the lower end of estimates for the age of the Universe based upon stellar evolution and nucleocosmochronology.

Now consider the case of a radiation-dominated Universe. In terms of \( \Omega_0 \) and \( z \),

\[
t = H_0^{-1} \frac{\sqrt{\Omega_0(1 + z)^2} - \sqrt{\Omega_0(1 + z)^2 - \Omega_0 + 1}}{(\Omega_0 - 1)(1 + z)}.
\]  
(3.28)

The present age of a radiation-dominated Universe is given by the above expression with \( z = 0 \)

\[
t_0 = H_0^{-1} \frac{\sqrt{\Omega_0} - 1}{\Omega_0 - 1}.
\]  
(3.29)

For \( \Omega_0 \) not too different from 1, (3.28) can be expanded to give

\[
t_0 \simeq \frac{1}{2} H_0^{-1} \left[ 1 - \frac{1}{4} (\Omega_0 - 1) + \cdots \right].
\]  
(3.30)

For \( (1 + z) \gg \Omega_0^{-1} \), (3.28) reduces to

\[
t \simeq \frac{1}{2} (1 + z)^{-2} H_0^{-1} \Omega_0^{-1/2}.
\]  
(3.31)

For both the matter-dominated and radiation-dominated cases, the present age of the Universe is shown in Fig. 3.1 as a function of \( \Omega_0 \). In both cases, \( H_0 t_0 \equiv f(\Omega_0) \) is a decreasing function of \( \Omega_0 \). As \( \Omega_0 \to 0 \), \( f(\Omega_0) \to 1 \). If \( \Omega_0 = 1 \), \( f(\Omega_0) = 2/3 \) for the matter-dominated case and \( f(\Omega_0) = 1/2 \) for the radiation-dominated case. As \( \Omega_0 \to \infty \), \( f(\Omega_0) \to (\pi/2)\Omega_0^{-1/2} \) for the matter-dominated case and \( f(\Omega_0) \to \Omega_0^{-1/2} \) for the radiation-dominated case.

Finally, consider the age of a model Universe that is flat, and contains both matter and (positive) vacuum energy (equivalently, a cosmological constant). The present age of such a model is also easily computed,

\[
t_0 = \frac{2}{3} H_0^{-1} \Omega_0^{-1/2} \ln \left[ \frac{1 + \Omega_{\text{VAC}}^{1/2}}{(1 - \Omega_{\text{VAC}}^{1/2})} \right]
\]  
(3.32)

where \( \Omega_{\text{VAC}} = \rho_{\text{VAC}}/\rho_C \) and by assumption \( \Omega_{\text{VAC}} + \Omega_{\text{MATTER}} = 1 \). The present age of such a model universe is shown in Fig. 3.2. It is interesting to note that unlike previous models, a model Universe with \( \Omega_{\text{VAC}} \gtrsim 0.74 \)
is older than $H_0^{-1}$; this occurs because the expansion rate is accelerating. And of course in the limit $\Omega_{\text{VAC}} \to 1$, $t_0 \to \infty$.

As we will discuss in greater length in Chapters 7 and 8, there is no understanding at present for the absence of a cosmological constant; moreover, the naive expectation for such is $\rho_{\text{VAC}} \sim m_{\text{Pl}}^4$, a value which would result in an expansion rate which is about a factor of $10^{61}$ larger than that observed today. The problem of reconciling a "youthful" expansion age with other independent age determinations has at several times in the past led cosmologists to invoke a cosmological constant (and may again in the future). Lacking a fundamental understanding of what its value should be, we should probably keep an open mind to the possibility that $\rho_{\text{VAC}} \neq 0$.

As we discussed in Chapter 1, attempts to directly measure $\Omega_0$ probably only restrict it to the range zero to a few. The age of the present Universe provides a very powerful constraint to the value of $\Omega_0$, and to the present energy density of the Universe. To see this, consider a matter-dominated model. Its present age is $t_0 = H_0^{-1} f(\Omega_0) = 9.78 h^{-1} f(\Omega_0)$ Gyr. The function $f(\Omega_0)$ decreases monotonically with $\Omega_0$ as discussed above (and shown in Fig. 3.1). For fixed $h$, a model with larger $\Omega_0$ is younger. Using
this fact and an independent age determination one can then derive an upper limit to $\Omega_0$. Independent measures of the age of the Universe suggest that $t_0 = 10$ to 20 Gyr (see Chapter 1). Since a smaller value for $f(\Omega_0)$ (larger $\Omega_0$) can be compensated for by a lower value of $h$, we also need a lower limit to $h$ in order to constrain $\Omega_0$.

Taking $t_0 \geq 10$ Gyr and $h \geq 0.4$ (0.5), it follows from Fig. 3.1 that $\Omega_0$ must be less than 6.4 (3.1). And of course a larger lower bound to $t_0$, say 12 Gyr, results in a more stringent upper bound to $\Omega_0$: $\Omega_0 \leq 3.5 \ (h \geq 0.4)$, or $\Omega_0 \leq 1.5 \ (h \geq 0.5)$.

In constraining the mass density contributed by a relic particle species,

$$\rho_X = \Omega_X \rho_C = \Omega_X h^2 1.88 \times 10^{-29} \text{g cm}^{-3}, \quad (3.33)$$

it proves useful to have a constraint to $\Omega_0 h^2$. By considering the age of the Universe once again, one can derive a more stringent constraint than the one that simply follows by using the individual constraints to $\Omega_0$ and $h$. Using again the fact that $t_0 = 9.78h^{-1}f(\Omega_0)$ Gyr, and defining $t_{10}$ by $t_0 = t_{10} \times 10^{10}$ yr, we have

$$\Omega_0 f^2(\Omega_0) = \Omega_0 h^2 (t_0/9.78 \text{Gyr})^2 \quad (3.34)$$

$$0.956 \Omega_0 f^2(\Omega_0)/t_{10}^2 \geq \Omega_0 h^2. \quad (3.35)$$

The function $\Omega_0 f^2(\Omega_0)$ increases monotonically, and is bounded from above by its value at $\Omega_0 = \infty$: $\Omega_0 f^2(\Omega_0) \to \pi^2/4 \simeq 2.47$ as $\Omega_0 \to \infty$. Irrespective of the value of $h$, if $t_{10} \geq 1$, then $\Omega_0 h^2 \leq 2.4$. Using our limited knowledge of $h$, we can do considerably better. Taking $h$ to be greater than 0.4 (0.5), we find that $\Omega_0 h^2 \leq 1$ (0.8); see Fig. 3.3.

One of the routine duties of an early-Universe cosmologist is to compute the present mass density contributed by some massive stable particle species hypothesized by a particle physicist down the hall, and to determine whether such a particle is at odds with the standard cosmology. In so doing, it is very convenient to express the mass density contributed by the particle species “$X$” in terms of $\rho_C$: $\rho_X = 1.88 \times 10^{-29} (\Omega_X h^2) \text{ g cm}^{-3}$. The best limit to $\rho_X$ follows from the age constraint discussed above: $\Omega_X h^2 \leq \Omega_0 h^2 \leq 1$. If the computed value of $\Omega_X h^2$ exceeds unity, it is often—and incorrectly—stated that the relic is forbidden because it would “overclose” the Universe. The existence of a cosmological relic cannot, of course, modify the geometry of the Universe, i.e., change an open, infinite Universe with $\Omega < 1$ to a closed, finite Universe with $\Omega > 1$. Rather, the
existence of a relic with $\Omega_X h^2 > 1$ would lead to a Universe that, at a temperature of 2.75 K, would have a higher mass density, larger expansion rate (Hubble constant), and smaller age than a Universe without relic $X$'s, because it would have become matter dominated at an earlier epoch. To illustrate, consider a flat model where the relic density of $X$'s at present is such that $\Omega_X h^2 = \beta^2 \gg 1$. Since a flat model is characterized by $\Omega = 1$ whether $X$'s are present or not, such a Universe must have $h^2 = \beta^2 \gg 1$. This would lead to a Hubble constant of $100 \beta$ km sec$^{-1}$Mpc$^{-1} \gg 100$ km sec$^{-1}$Mpc$^{-1}$, in contradiction to the observed value.

The equations previously derived for $t(z)$ can be inverted to give the scale factor $R$ as a function of $t$. To begin, ignore spatial curvature ($k = 0$) and consider the simple case in which the rhs of the Friedmann equation is dominated by a fluid whose pressure is given by $p = w \rho$. Then it follows that

$$\rho \propto R^{-3(1+w)}$$  \hspace{1cm} (3.36)

$$R \propto t^{2/3(1+w)}$$  \hspace{1cm} (3.37)

which leads to the familiar results: $R \propto t^{1/2}$ for $w = 1/3$ (RD); $R \propto t^{2/3}$
for \( w = 0 \) (MD); \( R \propto \exp(H_0 t) \) for \( w = -1 \) (vacuum dominated); and 
\( R \propto t \) for \( w = -1/3 \), which corresponds to a curvature-dominated model 
\( (H^2 \propto R^{-2}) \).

Now consider a matter-dominated model with arbitrary \( \Omega_0 > 1 \). While 
\( R(t) \) cannot be given in closed form, \( R \) and \( t \) can be represented parametrically in terms of the "development" angle \( \theta \):

\[
\frac{R(t)}{R_0} = (1 - \cos \theta) \frac{\Omega_0}{2(\Omega_0 - 1)} \quad (3.38)
\]

\[
H_0 t = (\theta - \sin \theta) \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}. \quad (3.39)
\]

The scale factor \( R(t) \) increases from zero (at \( t = \theta = 0 \)) to its maximum value

\[
\frac{R_{\text{MAX}}}{R_0} = \frac{\Omega_0}{(\Omega_0 - 1)} \quad (3.40)
\]

\[
H_0 t_{\text{MAX}} = \frac{\pi}{2} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \quad (3.41)
\]

\[
\theta_{\text{MAX}} = \pi \quad (3.42)
\]

and then back to zero (at \( \theta = 2\pi, t = 2t_{\text{MAX}} \)). The maximum value 
of \( R \) obtains when \( 8\pi G \rho/3 = k/R^2 \) and the expansion rate \( H \) vanishes. 
Further expansion would result in a negative value for \( H^2 \), so recollapse must follow.

Now consider a matter-dominated model with \( \Omega_0 < 1 \). In this case the 
development angle is imaginary, \( \theta = i\psi \):

\[
\frac{R(t)}{R_0} = (\cosh \psi - 1) \frac{\Omega_0}{2(1 - \Omega_0)^{3/2}}, \quad (3.43)
\]

\[
H_0 t = (\sinh \psi - \psi) \frac{\Omega_0}{2(1 - \Omega_0)^{3/2}}. \quad (3.44)
\]

Here, of course, \( R(t) \) increases without limit.

The evolution of \( R(t) \) for both the open, closed, and flat FRW models 
is shown in Fig. 3.4.

Finally, consider a model with comparable contributions to the energy 
density from both matter and radiation, but with negligible curvature: To
a very good approximation this corresponds to the Universe around the
time of matter–radiation equality \( R \equiv R_{EQ} \simeq 4 \times 10^{-5} (\Omega_0 h^2)^{-1} R_0 \). It is
simple to obtain the following expression for the time as a function of the
scale factor (inverting this expression is not so simple):

\[
t/t_{EQ} = [(R/R_{EQ} - 2)(R/R_{EQ} + 1)^{1/2} + 2]/(2 - \sqrt{2})
\] (3.45)

where the epoch of equal matter and radiation densities is denoted by
subscript EQ, and \( t_{EQ} = 4(\sqrt{2} - 1)H_{EQ}^{-1}/3 \). As they must be, the limiting
forms of (3.45) are: \( R \propto t^{1/2} \) (\( t \ll t_{EQ} \)), and \( R \propto t^{2/3} \) (\( t \gg t_{EQ} \)).

### 3.3 Equilibrium Thermodynamics

Today the radiation, or relativistic particles, in the Universe is comprised
of the 2.75 K microwave photons, and the 3 cosmic seas of 1.96 K relic
neutrinos. Because the early Universe was to a good approximation in
thermal equilibrium, there should have been other relativistic particles
present, with comparable abundances. Before going on to discuss the early
radiation-dominated phase, we will quickly review some basic thermodynamics.

The number density \( n \), energy density \( \rho \), and pressure \( p \) of a dilute, weakly-interacting gas of particles with \( g \) internal degrees of freedom is given in terms of its phase space distribution (or occupancy) function \( f(\vec{p}) \):

\[
n = \frac{g}{(2\pi)^3} \int f(\vec{p}) d^3p
\]

\[
\rho = \frac{g}{(2\pi)^3} \int E(\vec{p}) f(\vec{p}) d^3p
\]

\[
p = \frac{g}{(2\pi)^3} \int \frac{|\vec{p}|^2}{3E} f(\vec{p}) d^3p
\]

where \( E^2 = |\vec{p}|^2 + m^2 \). For a species in kinetic equilibrium the phase space occupancy \( f \) is given by the familiar Fermi-Dirac or Bose-Einstein distributions,

\[
f(\vec{p}) = \frac{[\exp((E - \mu)/T) + 1]}{1}
\]

where \( \mu \) is the chemical potential of the species, and here and throughout +1 pertains to Fermi-Dirac species and -1 to Bose-Einstein species. Moreover, if the species is in chemical equilibrium, then its chemical potential \( \mu \) is related to the chemical potentials of other species with which it interacts. For example, if the species (denoted by \( i \)) interacts with species \( j, k, l \),

\[
i + j \longleftrightarrow k + l
\]

then \( \mu_i + \mu_j = \mu_k + \mu_l \), whenever chemical equilibrium holds. We will return to the question of when local thermodynamic equilibrium (or, LTE) holds later.

From the equilibrium distributions, it follows that the number density \( n \), energy density \( \rho \), and pressure \( p \) of a species of mass \( m \) with chemical potential \( \mu \) at temperature \( T \) is

\[
\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp\left[\frac{(E - \mu)/T}{\pm 1}\right]} \frac{E \, dE}{dE}
\]

\[
n = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp\left[\frac{(E - \mu)/T}{\pm 1}\right]} \frac{E \, dE}{dE}
\]
\[ p = \frac{g}{6\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{3/2}}{\exp[(E - \mu)/T] + 1} \, dE. \] (3.51)

In the relativistic limit \((T \gg m)\), for \(T \gg \mu\),

\[ \rho = \begin{cases} 
\frac{\pi^2}{30} g T^4 & \text{(BOSE)} \\
\frac{7}{8} \frac{\pi^2}{30} g T^4 & \text{(FERMI)}
\end{cases} \]

\[ n = \begin{cases} 
\frac{\zeta(3)}{\pi^2} g T^3 & \text{(BOSE)} \\
\frac{3}{4} \frac{\zeta(3)}{\pi^2} g T^3 & \text{(FERMI)},
\end{cases} \]

\[ p = \rho/3 \] (3.52)

while for degenerate fermions \((\mu \gg T)\)

\[ \rho = \frac{1}{8\pi^2} g \mu^4 \]

\[ n = \frac{1}{6\pi^2} g \mu^3 \]

\[ p = \frac{1}{24\pi^2} g \mu^4. \] (3.53)

Here \(\zeta(3) = 1.20206...\) is the Riemann zeta function of 3. For a Bose-Einstein species \(\mu > 0\) indicates the presence of a Bose condensate, which must be treated separately from the other modes. For relativistic bosons or fermions with \(\mu < 0\) and \(|\mu| < T\), it follows that

\[ n = \exp(\mu/T)(g/\pi^2)T^3 \]

\[ \rho = \exp(\mu/T)(3g/\pi^2)T^4 \]

\[ p = \exp(\mu/T)(g/\pi^2)T^4. \] (3.54)

In the non-relativistic limit \((m \gg T)\) the number density, energy density and pressure are the same for Bose and Fermi species

\[ n = g \left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[-(m - \mu)/T\right] \]

\[ \rho = mn \]

\[ p = nT \ll \rho. \] (3.55)

For a nondegenerate, relativistic species, the average energy per particle
3.3 Thermodynamics

is

\[ \langle E \rangle \equiv \frac{\rho}{n} = \left[ \frac{\pi^4}{30 \zeta(3)} \right] T \simeq 2.701 T \quad (\text{BOSE}) \]

\[ \langle E \rangle \equiv \frac{\rho}{n} = \left[ \frac{7\pi^4}{180 \zeta(3)} \right] T \simeq 3.151 T \quad (\text{FERMI}). \quad (3.56) \]

For a degenerate, relativistic fermion species,

\[ \langle E \rangle \equiv \frac{\rho}{n} = \frac{3}{4} \mu. \quad (3.57) \]

For a non-relativistic species, \( \langle E \rangle = m + (3/2)T. \)

The excess of a fermion species over its antiparticle is often of interest, and is straightforward to compute in the relativistic and non-relativistic limits. Assuming that \( \mu_+ = -\mu_- \) (true if reactions like \( \text{particle} + \text{antiparticle} \leftrightarrow \gamma + \gamma \) are occurring rapidly), the net fermion number density is

\[ n_+ - n_- = \frac{g}{2\pi^2} \int_m^\infty E (E^2 - m^2)^{1/2} dE \times \left[ \frac{1}{1 + \exp[(E - \mu)/T]} - \frac{1}{1 + \exp[(E + \mu)/T]} \right] \]

\[ = \frac{g T^3}{6\pi^2} \left[ \frac{\mu}{T} + \left( \frac{\mu}{T} \right)^3 \right] \quad (T \gg m) \]

\[ = 2g (mT/2\pi)^{3/2} \sinh(\mu/T) \times \exp(-m/T) \quad (T \ll m). \quad (3.58) \]

The total energy density and pressure of all species in equilibrium can be expressed in terms of the photon temperature \( T \)

\[ \rho_R = T^4 \sum_{i=\text{all species}} \left( \frac{T_i}{T} \right)^4 \frac{g_i}{2\pi^2} \frac{1}{x_i} \frac{\int x_i^\infty \frac{(u^2 - x_i^2)^{1/2} u^2 du}{\exp(u - y_i) \pm 1}}{\exp(u - y_i) \pm 1} \quad (3.59) \]

\[ p_R = T^4 \sum_{i=\text{all species}} \left( \frac{T_i}{T} \right)^4 \frac{g_i}{6\pi^2} \frac{1}{x_i} \frac{\int x_i^\infty \frac{(u^2 - x_i^2)^{3/2} du}{\exp(u - y_i) \pm 1}}{\exp(u - y_i) \pm 1} \quad (3.60) \]

where \( x_i \equiv m_i/T, y_i \equiv \mu_i/T \), and we have taken into account the possibility that the species \( i \) may have a thermal distribution, but with a different temperature than that of the photons.
Since the energy density and pressure of a non-relativistic species (i.e., one with mass \( m \gg T \)) is exponentially smaller than that of a relativistic species (i.e., one with mass \( m \ll T \)), it is a very convenient and good approximation to include only the relativistic species in the sums for \( \rho_R \) and \( p_R \), in which case the above expressions greatly simplify:

\[
\rho_R = \frac{\pi^2}{90} g_* T^4, \\
p_R = \rho_R/3 = \frac{\pi^2}{90} g_* T^4,
\]

where \( g_* \) counts the total number of effectively massless degrees of freedom (those species with mass \( m_i \ll T \)), and

\[
g_* = \sum_{i = \text{bosons}} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i = \text{fermions}} g_i \left( \frac{T_i}{T} \right)^4. 
\]

(3.61)

The relative factor of \( 7/8 \) accounts for the difference in Fermi and Bose statistics. Of course, it is a straightforward matter to obtain an exact expression for \( g_*(T) \) from (3.59).\(^5\) Note also that \( g_* \) is a function of \( T \) since the sum runs over only those species with mass \( m_i \ll T \). For \( T \ll \text{MeV} \), the only relativistic species are the 3 neutrino species (assuming that they are very light) and the photon; since \( T_{\nu} = (4/11)^{1/3} T_\gamma \) (see below), \( g_* (\ll \text{MeV}) = 3.36 \). For \( 100 \text{ MeV} \gtrsim T \gtrsim 1 \text{ MeV} \), the electron and positron are additional relativistic degrees of freedom and \( T_\nu = T_\gamma \); \( g_* = 10.75 \). For \( T \gtrsim 300 \text{ GeV} \), all the species in the standard model—8 gluons, \( W^\pm Z^0 \), 3 generations of quarks and leptons, and 1 complex Higgs doublet—should have been relativistic; \( g_* = 106.75 \). The dependence of \( g_*(T) \) upon \( T \) is shown in Fig. 3.5.

During the early radiation-dominated epoch \( (t \lesssim 4 \times 10^{10} \text{ sec}) \rho \simeq \rho_R \); and further, when \( g_* \simeq \text{const} \), \( p_R = \rho_R/3 \) (i.e., \( w = 1/3 \)) and \( R(t) \propto t^{1/2} \). From this it follows

\[
H = 1.66 g_*^{1/2} \frac{T^2}{m_{pl}} \\
t = 0.301 g_*^{-1/2} m_{pl} \frac{T^2}{T^2} \sim \left( \frac{T}{\text{MeV}} \right)^{-2} \text{ sec}. 
\]

(3.63)

\(^5\)If the contribution to \( \rho \) and \( p \) of non-relativistic or semi-relativistic species are significant the \( g_*(T) \) for \( \rho \) and \( p \) are not equal; see Fig. 3.5.
Fig. 3.5: The evolution of $g_s(T)$ as a function of temperature in the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ theory.

3.4 Entropy

Throughout most of the history of the Universe (in particular the early Universe) the reaction rates of particles in the thermal bath, $\Gamma_{\text{int}}$, were much greater than the expansion rate, $H$, and local thermal equilibrium (LTE) should have been maintained. In this case the entropy per comoving volume element remains constant. The entropy in a comoving volume provides a very useful fiducial quantity during the expansion of the Universe.

In the expanding Universe, the second law of thermodynamics, as applied to a comoving volume element of unit coordinate volume\(^6\) and physical volume $V = R^3$, implies that

$$TdS = d(\rho V) + pdV = d[(\rho + p)V] - V dp,$$  \hspace{1cm} (3.64)

where $\rho$ and $p$ are the equilibrium energy density and pressure. Moreover,

\(^6\)For simplicity, unless otherwise noted, we will take our comoving volume to be of unit coordinate volume.
the integrability condition,

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T}$$  \hspace{1cm} (3.65)

relates the energy density and pressure:

$$T \frac{dp}{dT} = \rho + p,$$  \hspace{1cm} (3.66)

or equivalently\(^{7}\)

$$dp = \frac{\rho + p}{T} dT.$$  \hspace{1cm} (3.67)

If we substitute (3.67) into (3.64), it follows that

$$dS = \frac{1}{T} d[(\rho + p) V] - (\rho + p) V \frac{dT}{T^2} = d\left[\frac{(\rho + p) V}{T} + \text{const}\right].$$  \hspace{1cm} (3.68)

That is, up to an additive constant, the entropy per comoving volume is \(S = R^3 (\rho + p)/T\). Recall that the first law (energy conservation) can be written as

$$d[(\rho + p) V] = V dp.$$  \hspace{1cm} (3.69)

Substituting (3.67) into (3.69), it follows that

$$d\left[\frac{(\rho + p) V}{T}\right] = 0.$$  \hspace{1cm} (3.70)

This result implies that in thermal equilibrium, the entropy per comoving volume, \(S\), is conserved.\(^{8}\)

It is useful to define the entropy density \(s\)

$$s \equiv \frac{S}{V} = \frac{\rho + p}{T}.$$  \hspace{1cm} (3.71)

\(^{7}\)Note, as it must, (3.67) follows directly from the equilibrium expressions for the pressure and energy density.

\(^{8}\)Here we have assumed that all chemical potentials are zero—a very good approximation, as all evidence indicates that \(|\mu| \ll T\). It is straightforward to include a species with a chemical potential; in this case \(TdS = d(\rho V) + pdV - \mu d(nV)\), and \(S = R^3 (\rho + p - \mu n)/T\).
The entropy density is dominated by the contribution of relativistic particles, so that to a very good approximation,

$$s = \frac{2\pi^2}{45} g_* s T^3,$$

(3.72)

where

$$g_* s = \sum_{i=bosons} g_i \left( \frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{i=fermions} g_i \left( \frac{T_i}{T} \right)^3.$$

(3.73)

For most of the history of the Universe all particle species had a common temperature, and $g_* s$ can be replaced by $g_*$. Note also that $s$ is proportional to the number density of relativistic particles, and that in particular $s$ is related to the photon number density, $s = 1.80 g_* s n_\gamma$, where $n_\gamma$ is the number density of photons. Today $s = 7.04 n_\gamma$. Since $g_* s$ is a function of temperature, $s$ and $n_\gamma$ cannot always be used interchangeably.

Conservation of $S$ implies that $s \propto R^{-3}$, and therefore that $g_* s T^3 R^3$ remains constant as the Universe expands. The first fact, that $s \propto R^{-3}$, implies that the physical size of a comoving volume element $\propto R^3 \propto s^{-1}$. Thus the number of some species in a comoving volume, $N \equiv R^3 n$, is equal to the number density of that species divided by $s$:

$$N \equiv n / s.$$  

(3.74)

For a species in thermal equilibrium

$$N = \frac{45 \zeta(3) g}{2\pi^4 g_* s} \quad T \gg m, \mu$$

$$= \frac{45 g}{4 \sqrt{2\pi^5 g_* s}} (m/T)^{3/2} \exp(-m/T + \mu/T) \quad T \ll m.$$  

(3.75)

If the number of a given species in a comoving volume is not changing, i.e., particles of that species are not being created or destroyed, then $N = n / s$ remains constant. The number of a species in thermal equilibrium in a comoving volume is shown in Fig. 3.6 for $\mu = 0$.

As an example of the utility of the ratio $n / s$, consider baryon number; the baryon number in a comoving volume is

$$\frac{n_B}{s} \equiv \frac{n_b - n_\bar{b}}{s}.$$  

(3.76)
Fig. 3.6: The equilibrium abundance of a species in a comoving volume element, \( N = n/s \). Since both \( n_\gamma \) and \( s \) vary as \( T^3 \), \( N \) is also proportional to \( n/n_\gamma \).

So long as baryon number nonconserving interactions (if such exist in nature) are occurring very slowly, the baryon number in a comoving volume, \( n_B/s \), is conserved. Although \( \eta = n_B/n_\gamma = 1.8g_s(n_B/s) \), the baryon number-to-photon ratio does not remain constant with time because \( g_s \) changes. During the era of \( e^\pm \) annihilation, the number of photons per comoving volume, \( N_\gamma = R^3n_\gamma \), increases by a factor of 11/4, so that \( \eta \) decreases by the same factor. After the time of \( e^\pm \) annihilations, however, \( g_s \) is constant, and \( \eta \simeq 7n_B/s \) and \( n_B/s \) can be used interchangeably.

The second fact, that \( S = g_sT^3R^3 = \text{const} \), implies that the temperature of the Universe evolves as

\[
T \propto g_s^{-1/3}R^{-1}. \tag{3.77}
\]

Whenever \( g_s \) is constant, the familiar result, \( T \propto R^{-1} \), obtains. The factor of \( g_s^{-1/3} \) enters because whenever a particle species becomes non-relativistic and disappears (see Fig. 3.6), its entropy is transferred to the other relativistic particle species still present in the thermal plasma, caus-
ing $T$ to decrease slightly less slowly.\footnote{We have been careful to distinguish between $g_\ast$ and $g_{\ast S}$ in this Section. While today $g_\ast \neq g_{\ast S}$, earlier than 1 sec the difference should have been small, and henceforth, we will not distinguish between the two.}

Massless particles that are decoupled from the heat bath will not share in the entropy transfer as the temperature drops below the mass threshold of a species; instead, the temperature of a massless decoupled species scales as $T \propto R^{-1}$ as we will now show. Consider a massless particle species initially in LTE which decouples at time $t_D$, temperature $T_D$, when the scale factor was $R_D$. The phase-space distribution at decoupling is given by the equilibrium distribution

$$f(\vec{p}, t_D) = [\exp(E/T_D) \pm 1]^{-1}.$$  \hspace{1cm} (3.78)

After decoupling, the energy of each massless particle is red shifted by the expansion of the Universe: $E(t) = E(t_D)(R(t_D)/R(t))$. In addition, the number density of particles decreases due to the expansion of the Universe: $n \propto R^{-3}$. As a result, the phase space distribution function, $f(\vec{p}) = d^3n/d^3p$, at time $t$ will be precisely that of a species in LTE with temperature $T(t) = T_D R_D/R(t)$:

$$f(\vec{p}, t) = f(\frac{\vec{p} R}{R_D}, t_D) = \left[\exp\left(\frac{ER}{R_D T_D}\right) \pm 1\right]^{-1}$$
$$= [\exp(E/T) \pm 1]^{-1}.$$ \hspace{1cm} (3.79)

Thus the distribution function for a massless particle species remains self-similar as the Universe expands, with the temperature red shifting as $R^{-1}$

$$T = T_D \frac{R_D}{R} \propto R^{-1} \hspace{1cm}\text{DECOUPLED - MASSLESS,}$$  \hspace{1cm} (3.80)

and not as $g_\ast^{-1/3}R^{-1} \neq g_{\ast S}^{-1/3}R^{-1}$ which holds for the particle species remaining in equilibrium.

Next, consider the evolution of the phase space distribution of a massive, non-relativistic ($m \gg T_D$) particle species that was in LTE, and then decouples (when $t = t_D$, $T = T_D$, and $R = R_D$). The momentum of each particle red shifts as the Universe expands: $|\vec{p}(t)| = |\vec{p}_D|R_D/R$; from which it follows that the kinetic energy of each particle red shifts as $R^{-2}$: $E_K(t) = E_K(t_D)R^2/R^2$. Further, the number density of particles
decreases as $R^{-3}$. Owing to both of these effects, such a decoupled species will have precisely an equilibrium phase space distribution characterized by temperature $T$,

$$T = T_D \left( \frac{R_D}{R} \right)^2 \propto R^{-2} \quad \text{DECOUPL ED - MASSIVE,} \quad (3.81)$$

and chemical potential

$$\mu(t) = m + (\mu_D - m)T(t)/T_D. \quad (3.82)$$

The chemical potential must vary in this way to insure that the number density of particles scales with the expansion as $R^{-3}$.

So we see that a decoupled species that is either highly relativistic ($T_D \gg m$) or highly non-relativistic ($m \gg T_D$) at decoupling maintains an equilibrium distribution; the former characterized by a temperature $\propto R^{-1}$, and the latter by a temperature $\propto R^{-2}$. For a species that decouples when it is semi-relativistic, $T_D \sim m$, the phase space distribution does not maintain an equilibrium distribution in the absence of interactions.

### 3.5 Brief Thermal History of the Universe

In the strictest mathematical sense it is not possible for the Universe to be in thermal equilibrium, as the FRW cosmological model does not possess a time-like Killing vector. For practical purposes, however, the Universe has for much of its history been very nearly in thermal equilibrium. Needless to say the departures from equilibrium have been very important—without them, the past history of the Universe would be irrelevant, as the present state would be merely that of a system at 2.75 K, very uninteresting indeed! The key to understanding the thermal history of the Universe is the comparison of the particle interaction rates and the expansion rate. Ignoring the temperature variation of $g_*$ for this discussion, $T \propto R^{-1}$ and the rate of change of the temperature $\dot{T}/T$ is just set by the expansion rate: $\dot{T}/T = -H$. So long as the interactions necessary for particle distribution functions to adjust to the changing temperature are rapid compared to the expansion rate, the Universe will, to a good approximation, evolve through a succession of nearly thermal states with temperature decreasing as $R^{-1}$. A useful rule of thumb is that a reaction is occurring rapidly enough to maintain thermal distributions when $\Gamma \gtrsim H$, where $\Gamma$ is the interaction
rate per particle, \( \Gamma \equiv n\sigma|v| \). Here \( n \) is the number density of target particles and \( \sigma|v| \) is the cross section for interaction times relative velocity (appropriately averaged).

This criterion is simple to understand. Suppose, as is very often the case, \( \Gamma \propto T^n \); then the number of interactions a species has from time \( t \) onward is given by

\[
N_{\text{int}} = \int_t^\infty \Gamma(t')dt'.
\] (3.83)

Taking the Universe to be radiation dominated, it follows that \( N_{\text{int}} = (\Gamma/H)|_t/(n - 2) \): For \( n > 2 \), a particle interacts less than one time subsequently to the time when \( \Gamma \simeq H \).

We remind the reader that \( \Gamma < H \) is not a sufficient condition for a departure from thermal equilibrium to occur: A massless, non-interacting species once in thermal equilibrium will forever maintain an equilibrium distribution with \( T \propto R^{-1} \). In order for a departure from equilibrium to develop, the rate for some reaction crucial to maintaining equilibrium must remain less than \( H \).

The correct way to evolve particle distributions is to integrate the Boltzmann equation; this approach will be used in Chapter 5 to precisely calculate the relic density of a stable particle species which decouples, and in Chapter 6 to compute the baryon asymmetry of the Universe. For the moment we will use \( \Gamma > H (\Gamma < H) \) as the criterion for whether or not a species is coupled to (decoupled from) the thermal plasma in the Universe.

To get a rough understanding of the decoupling of a particle species in the expanding Universe, consider two types of interactions: (i) interactions mediated by a massless gauge boson, e.g., the photon; (ii) interactions mediated by a massive gauge boson, e.g., a \( W^\pm \) or \( Z^0 \) boson below the scale of electroweak symmetry breaking (\( T \lesssim 300 \text{ GeV} \)). In the first case the cross section for a \( 2 \leftrightarrow 2 \) scattering of relativistic particles with significant momentum transfer is \( \sigma \sim \alpha^2/T^2 \) (\( g = \sqrt{4\pi\alpha} = \text{gauge coupling strength} \)). In the second case, for \( T \lesssim m_X \), the corresponding cross section is \( \sigma \sim G^2_X T^2 \) where \( m_X \) is the mass of the gauge boson, \( G_X \sim \alpha/m_X^2 \), and the \( m_X^{-2} \) factor results from the propagator of the massive gauge boson. For \( T \gg m_X \), the cross section is the same as that for massless gauge boson exchange.

For interactions mediated by massless gauge bosons \( \Gamma \sim n\sigma|v| \sim \alpha^2 T \); during the radiation-dominated epoch \( H \sim T^2/m_{Pl} \), so that \( \Gamma/H \sim \alpha^2 m_{Pl}/T \).\(^{10} \) Therefore, for \( T \lesssim \alpha^2 m_{Pl} \sim 10^{16} \text{ GeV} \) or so, such reactions

\(^{10}\)We will assume the target particles are relativistic and in LTE, so that \( n \sim T^3 \).
are occurring rapidly, while for \( T \gtrsim \alpha^2 m_{Pl} \sim 10^{16} \text{ GeV} \), such reactions are effectively "frozen out." Now consider interactions mediated by massive gauge bosons: \( \Gamma \sim n\sigma |v|^2 \sim G_X^2 T^5 \) and \( \Gamma / H \sim G_X^2 m_{Pl} T^3 \). Thus for \( m_X \gtrsim T \gtrsim G_X^{-2/3} m_{Pl}^{-1/3} \sim (m_X/100 \text{ GeV})^{4/3} \text{ MeV} \), such reactions are occurring rapidly, while for \( T \lesssim (m_X/100 \text{ GeV})^{4/3} \text{ MeV} \) such reactions have effectively frozen out.

We should emphasize that for \( T \gtrsim \alpha^2 m_{Pl} \sim 10^{16} \text{ GeV} \) all perturbative interactions should be frozen out and ineffective in maintaining or establishing thermal equilibrium. Thus the known interactions—plus any new interactions arising from grand unification—are not capable of thermalizing the Universe at temperatures greater than \( 10^{16} \text{ GeV} \), corresponding to times earlier than about \( 10^{-38} \text{ sec} \). Perhaps there exist other, as of yet unknown, interactions that can thermalize the Universe at these earliest times (e.g., "strong" gravitational interactions). We should keep in mind the fact that the Universe may not have been in thermal equilibrium during its earliest epoch (\( T \gtrsim 10^{16} \text{ GeV} \)).

Fig. 3.7 provides a brief summary of the thermal history of the Universe, based upon extrapolating our present knowledge of the Universe and particle physics back to the Planck epoch (\( t \sim 10^{-43} \text{ sec} \) and \( T \sim 10^{19} \text{ GeV} \)), the point at which quantum corrections to general relativity should render it invalid. At the earliest times the Universe was a plasma of relativistic particles, including the quarks, leptons, gauge bosons, and Higgs bosons. If current ideas are correct, a number of spontaneous symmetry breaking (or SSB) phase transitions should take place during the course of the early history of the Universe. They include the grand unification (or GUT) phase transition at a temperature of \( 10^{14} \) to \( 10^{16} \text{ GeV} \), and the electroweak SSB phase transition at a temperature of about 300 GeV. During these SSB phase transitions some of the gauge bosons and other particles acquire mass via the Higgs mechanism and the full symmetry of the theory is broken to a lower symmetry. Subsequent to the phase transition the interactions mediated by the \( X \) bosons which acquire mass will be characterized by a coupling strength \( G_X \), and particles which only interact via such interactions will decouple from the thermal plasma at \( T \sim G_X^{-2/3} m_{Pl}^{-1/3} \). SSB phase transitions will be discussed in Chapter 7.

At a temperature of about 100 to 300 MeV (\( t \sim 10^{-5} \text{ sec} \)) the Universe should undergo a transition associated with chiral symmetry breaking and color confinement, after which the strongly-interacting particles are color-singlet-quark-triplet states (baryons) and color-singlet-quark-antiquark states (mesons). While we will not discuss the quark/hadron
Fig. 3.7: The complete history of the Universe.
transition, the details and the nature (1st order, 2nd order, etc.) of this transition are of some cosmological interest, as local inhomogeneities in the baryon number density may be produced and could possibly affect the outcome of primordial nucleosynthesis (as discussed in the next Chapter).

The epoch of primordial nucleosynthesis follows when \( t \sim 10^{-2} \) to \( 10^2 \) sec and \( T \sim 10 \) to 0.1 MeV; the next Chapter will be devoted to a detailed discussion of this very important subject. At present, primordial nucleosynthesis is the earliest test of the standard cosmology. At a time of about \( 10^{11} \) sec the matter density becomes equal to that of the radiation. This marks the beginning of the current matter-dominated epoch and the start of structure formation. Structure formation will be addressed in Chapter 9. Finally, at time of about \( 10^{13} \) sec, the ions and electrons combine to form atoms, and the matter and radiation decouple, ending the long epoch of near thermal equilibrium that existed in the early Universe. The surface of last scattering for the microwave background radiation is the Universe itself at decoupling. We will now discuss some of these events in more detail.

- **Neutrino Decoupling:** The phenomenon of decoupling of a massless species is nicely illustrated by considering the decoupling of massless neutrinos. In the early Universe neutrinos are kept in equilibrium via reactions of the sort \( \bar{\nu} \nu \leftrightarrow e^+ e^- \), \( \nu_e \leftrightarrow \nu_e \), etc. The cross section for these weak interaction processes is given by \( \sigma \approx G_F^2 T^2 \), where \( G_F \) is the Fermi constant. The number density of massless particles is \( n \sim T^3 \), and so the interaction rate (per neutrino) is

\[
\Gamma_{\text{int}} = n\sigma|v| \sim G_F^2 T^5. \tag{3.84}
\]

The ratio of the interaction rate to the expansion rate is

\[
\frac{\Gamma_{\text{int}}}{H} \approx \frac{G_F^2 T^5}{T^2/m_{Pl}} \approx \left( \frac{T}{1 \text{ MeV}} \right)^3. \tag{3.85}
\]

At temperatures above 1 MeV, the interaction rate is greater than the expansion rate and neutrinos are in good thermal contact with the plasma. At temperatures below 1 MeV the interaction rate is less than the expansion rate and neutrino interactions are too weak to keep them in equilibrium: Thus at a temperature of order 1 MeV light neutrino species decouple from the plasma. Below 1 MeV the neutrino temperature \( T_\nu \) scales as \( R^{-1} \). Shortly after neutrino decoupling the temperature drops below the mass of the electron, and the entropy in \( e^\pm \) pairs is transferred to the photons, but not to the decoupled neutrinos. For \( T \gtrsim m_e \), the
3.5 Brief Thermal History

Fig. 3.8: The evolution of $T$ and $T_\nu$ through the epoch of $e^\pm$ annihilation.

Particle species in thermal equilibrium with photons include the photon ($g = 2$) and $e^\pm$ pairs ($g = 4$), for a value of $g_* = 11/2$. For $T \ll m_e$, only the photons are in equilibrium for a value of $g_* = 2$. For the particles in thermal equilibrium with the photons $g_* (RT)^3$ remains constant; therefore the value of $RT$ after $e^\pm$ annihilation must be larger than that before $e^\pm$ annihilation by a factor of the third-root of the ratio of $g_*$ before $e^\pm$ annihilation ($=11/2$) to $g_*$ after $e^\pm$ annihilation ($=2$). Thus the $e^\pm$ entropy transfer increases $(RT_\gamma)$ by a factor of $(11/4)^{1/3}$, while $(RT_\nu)$ remains constant. Therefore today the ratio of $T$ and $T_\nu$ should be

$$\frac{T}{T_\nu} = \left(\frac{11}{4}\right)^{1/3} = 1.40$$

(3.86)

which gives $T_\nu = 1.96$ K. The increase of $T$ relative to $T_\nu$ is shown in Fig. 3.8. Note that the decrease in $g_*$ does not lead to an actual increase in $T$, but rather causes $T$ to decrease less slowly than $R^{-1}$.

Using this result we can now compute the values of $g_*$ and $g_* S$ today
(assuming 3 massless neutrino species)

\[ g_\ast (\text{today}) = 2 + \frac{7}{8} \times 2 \times 3 \times \left( \frac{4}{11} \right)^{4/3} = 3.36, \quad (3.87) \]

\[ g_{\ast s}(\text{today}) = 2 + \frac{7}{8} \times 2 \times 3 \times \frac{4}{11} = 3.91. \quad (3.88) \]

Note that since \( T_\nu \neq T, g_\ast \neq g_{\ast s}. \) Also note that since the photon and neutrino species are decoupled their entropies are separately conserved (a fact which we implicitly used above). Using these results we can compute the present energy density and entropy density

\[ \rho_R = \frac{\pi^2}{30} g_\ast T^4 = 8.09 \times 10^{-34} \text{ g cm}^{-3} \]

\[ \Omega_R h^2 = 4.31 \times 10^{-5} \]

\[ s = \frac{2\pi^2}{45} g_{\ast s} T^3 \simeq 2970 \text{ cm}^{-3} \]

\[ n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3 = 422 \text{ cm}^{-3}. \quad (3.89) \]

assuming that \( T_0 = 2.75 \text{K}. \)

\textbf{Graviton Decoupling:} Another example of decoupling is that of gravitons. On purely dimensional grounds, the interaction rate for particles with only gravitational strength interactions should be \( \Gamma_{\text{int}} = n \sigma |v| \simeq G^2 T^6 \simeq T^6/m_{Pl}^4 \) (which follows by taking \( G_X = G_N \) and \( \alpha G \sim 1 \)). This will become less than the expansion rate, \( H \simeq T^2/m_{Pl} \), at temperatures less than about \( m_{Pl} \). At the Planck time, the contribution to \( g_\ast \) from the particles of the standard model is 106.75; of course, if current ideas about unification are correct, there are likely many more relativistic degrees of freedom at \( T \sim m_{Pl} \). In any case, if gravitons were in thermal equilibrium at the Planck epoch (or before) and then decoupled, their present temperature should be at most \( (3.91/106.75)^{1/3} T \simeq 0.91 \text{K} \), corresponding to a number density of less than about 15 cm\(^{-3}\).

\textbf{Matter–Radiation Equality:} If we define \( \rho_M \) as the total energy density in "matter" (i.e., in non-relativistic particles: baryons and whatever else), then today \( \rho_M = 1.88 \times 10^{-28} \Omega_0 h^2 \text{ g cm}^{-3} \), where \( \Omega_0 \) is the fraction of the critical density contributed by matter. Using (3.89), and the fact that \( \rho_R/\rho_M \propto R_0/R = 1 + z \), it then follows that the red shift, time, and
temperature of equal matter and radiation energy densities are given by

\[ 1 + z_{\text{EQ}} \equiv \frac{R_0}{R_{\text{EQ}}} = 2.32 \times 10^4 \Omega_0 h^2 \]

\[ T_{\text{EQ}} = T_0(1 + z_{\text{EQ}}) = 5.50 \Omega_0 h^2 \text{ eV} \]

\[ t_{\text{EQ}} \simeq \frac{2}{3} H_0^{-1} \Omega_0^{-1/2}(1 + z_{\text{EQ}})^{-3/2} \]

\[ = 1.4 \times 10^3 (\Omega_0 h^2)^{-2} \text{ years.} \quad (3.90) \]

Note that the exact relationship for \( t_{\text{EQ}} \) [cf. (3.45)] results in a slightly different value: \( t_{\text{EQ}} = 0.39 H_0^{-1} \Omega_0^{-1/2}(1 + z_{\text{EQ}})^{-3/2} \).

- **Photon Decoupling and Recombination:** In the early Universe the matter and radiation were in good thermal contact, because of rapid interactions between the photons and electrons. However, eventually the density of free electrons became too low to maintain thermal contact and matter and radiation decoupled. Roughly speaking this occurs when \( \Gamma_\gamma \simeq H \), or equivalently when the mean free path of the photons, \( \lambda_\gamma \simeq \Gamma_\gamma^{-1} \), became larger than the Hubble distance, \( H^{-1} \).

The interaction rate of the photons is given by

\[ \Gamma_\gamma = n_e \sigma_T, \quad (3.91) \]

where \( n_e \) is the number density of free electrons, and \( \sigma_T \) is the Thomson cross section, \( \sigma_T = 6.65 \times 10^{-25} \text{ cm}^2 \). The equilibrium abundance of free electrons is determined by the Saha equation. The derivation of the Saha equation will be a useful warm-up exercise for the calculation of elemental abundances in nuclear statistical equilibrium, to be done in the next Chapter.

Let \( n_H, n_p, \) and \( n_e \) denote the number density of hydrogen, free protons, and free electrons, respectively. For simplicity we will ignore the one \(^4\text{He}\) nucleus per 10 protons and assume that all the baryons in the Universe are in the form of protons. The charge neutrality of the Universe implies \( n_p = n_e \), and baryon number conservation implies that \( n_B = n_p + n_H \). In thermal equilibrium, at temperatures less than \( m_i \)

\[ n_i = g_i \left( \frac{m_i T}{2\pi} \right)^{3/2} \exp \left( \frac{\mu_i - m_i}{T} \right), \quad (3.92) \]

where \( i = e, p, H \); \( m_i \) is the mass of species \( i \), and \( \mu_i \) is the chemical potential of \( i \). In chemical equilibrium, the process \( p + e \to H + \gamma \) guarantees
that $\mu_p + \mu_e = \mu_H$. The factor of $\mu_H$ in $n_H$ can be expressed in terms of $\mu_e$ and $\mu_p$, which, in turn, can be expressed in terms of $n_p$ and $n_e$:

$$n_H = \frac{g_H}{g_p g_e} n_p n_e \left( \frac{m_e T}{2\pi} \right)^{-3/2} \exp(B/T), \quad (3.93)$$

where $B$ is the binding energy of hydrogen, $B \equiv m_p + m_e - m_H = 13.6$ eV. In the pre-exponential factor we have set $m_H = m_p$. In terms of the total baryon number density the fractional ionization is

$$X_e \equiv \frac{n_p}{n_B}. \quad (3.94)$$

Using $g_p = g_e = 2$ and $g_H = 4$, and $n_B = \eta n_\gamma$, the equation for $n_H$ gives the equilibrium fractional ionization

$$\frac{1 - X_e^{eq}}{(X_e^{eq})^2} = \frac{4 \sqrt{2} \zeta(3)}{\sqrt{\pi}} \eta \left( \frac{T}{m_e} \right)^{3/2} \exp(B/T). \quad (3.95)$$

This is the Saha equation for the equilibrium ionization fraction.

The baryon-to-photon ratio $\eta$ is related to $\Omega_B h^2$ by $\eta = (\Omega_B h^2) 2.68 \times 10^{-8}$, and the temperature $T$ is related to the red shift $z$ by $T = (1 + z) 2.75$ K. Using these relations we have solved (3.95) for the equilibrium ionization for three values of $\Omega_B h^2$, with the results shown in Fig. 3.9. If we define recombination as the point when 90% of the electrons have combined with protons, we see from Fig. 3.9 that recombination occurred at a red shift between 1200 and 1400, depending on the value of $\Omega_B h^2$. Taking $1 + z = 1300$ for the red shift of recombination, the temperature at recombination is

$$T_{rec} = T_0(1 + z_{rec}) = 3575 \text{ K} = 0.308 \text{ eV}. \quad (3.96)$$

Assuming that the Universe was matter dominated at recombination, the age of the Universe at recombination is

$$t_{rec} = \frac{2}{3} H_0^{-1} \Omega_0^{-1/2} (1 + z_{rec})^{-3/2}$$

$$= 4.39 \times 10^{12} (\Omega_0 h^2)^{-1/2} \text{ sec}. \quad (3.97)$$

Note that recombination occurs at a temperature of about 0.3 eV, not at $T \sim B \sim 13.6$ eV. This is due to the small value of the prefactors.
to $\exp(B/T)$ in (3.95): The large entropy (small $\eta$) and $(T/m_e)^{3/2}$ factor result in $T_{rec}$ being much less than the binding energy of hydrogen.

The equilibrium ionization is only appropriate to use when equilibrium is maintained, i.e., when the reaction rate for $e + p \leftrightarrow H + \gamma$ is greater than the expansion rate. This question will be treated in detail in Chapter 5, where it will be shown that the equilibrium ionization is maintained for $(1+z) > 1100$, and that a residual ionization fraction of

$$X_\infty \simeq 3 \times 10^{-5} \Omega_0 / \Omega_B h$$

(3.98)

remains for $(1+z) \lesssim 1100$.

Using the equilibrium ionization, and the fact that the density of free electrons is $n_e = X_e n_B = X_e \eta n_\gamma \simeq X_e (\Omega_B h^2)(1+z)^3 1.13 \times 10^{-5} \text{ cm}^{-3}$, the photon mean free path can be found from (3.91), and compared to the age of the Universe, $t = (2/3)(1+z)^{-3/2} H_0^{-1} \Omega_0^{-1/2}$. The mean free path of the photons and the age of the Universe as a function of $(1+z)$ are shown in Fig. 3.10. The age of the Universe depends upon $\Omega_0$, while the mean free path depends on $\Omega_B$. Decoupling occurs when $\lambda_\gamma \sim \Gamma^{-1}_\gamma \simeq t \sim H^{-1}$.

The red shift at decoupling depends upon $\Omega_0$ and $\Omega_B$ (remember they
Fig. 3.10: The mean free path of the microwave photons (solid line), and the age of the Universe (dashed line) as a function of \((1 + z)\).

might well be different), but is somewhere in the range \(1 + z = 1100\) to 1200. If decoupling occurred at a red shift of \(1 + z \simeq 1100\), the temperature at decoupling was

\[
T_{\text{dec}} = T_0(1 + z_{\text{rec}}) = 3030 \, \text{K} = 0.26 \, \text{eV}. \tag{3.99}
\]

If the Universe was matter dominated at decoupling, the age of the Universe at decoupling was

\[
t_{\text{dec}} = \frac{2}{3} H_0^{-1} \Omega_0^{-1/2} (1 + z_{\text{dec}})^{-3/2} = 5.64 \times 10^{12} (\Omega_0 h^2)^{-1/2} \, \text{sec}. \tag{3.100}
\]

In finishing our discussion of “decoupling,” we want to emphasize that there are at least three distinct events occurring at this epoch: the recombination of matter, the decoupling of radiation, and the “freeze in” of a residual ionization. Fitting the results shown in Figs. 3.9 and 3.10 to simple analytic formulae, and stating the results obtained for the freeze in
of the residual ionization derived in Section 5.3, we find that these events occur at red shifts of

\[
1 + z_{\text{dec}} \approx 1100(\Omega_0/\Omega_B)^{0.018} \approx 1100 - 1200
\]
\[
1 + z_{\text{rec}} \approx 1380(\Omega_B h^2)^{0.023} \approx 1240 - 1380
\]
\[
1 + z_{\text{freeze}} \approx \frac{1180}{1 + 0.021 \ln(\Omega_0/\Omega_B)} \approx 1080 - 1180,
\]

where the red shift range corresponds to \( \Omega_B h^2 = 0.01 \) to 1 and \( \Omega_0/\Omega_B = 1 \) to 100, and we have assumed that the Universe is matter dominated at this epoch. For simplicity, and somewhat arbitrarily, we shall define the red shift of “decoupling/recombination/freeze in” to be \( 1 + z_{\text{dec}} \equiv 1100 \), corresponding to a temperature of 0.26 eV. We shall consistently use this value throughout the monograph.

- The Baryon Number of the Universe: The baryon number density is \( n_B \equiv n_b - n_\bar{b} \), where \( n_b \) (\( n_\bar{b} \)) is the baryon (antibaryon) number density. Today, all evidence suggests that there are very few antibaryons in the Universe, and the only baryons present are nucleons, so

\[
n_B = n_N = 1.13 \times 10^{-5} \ (\Omega_B h^2) \text{ cm}^{-3}.
\]

As discussed earlier, the ratio of the net baryon number to the entropy, \( n_B/s \), corresponds to the net baryon number in a comoving volume, a number that is conserved in the absence of baryon number violating reactions. Today that ratio is also the nucleon number in a comoving volume. Since that ratio serves to quantify the baryon number, we define it to be the baryon number of the Universe:

\[
B \equiv \frac{n_B}{s} = 3.81 \times 10^{-9} \ (\Omega_B h^2).
\]

Since the epoch of \( e^\pm \) annihilation the entropy density \( s \) and the photon number density \( n_\gamma \) have been related by a constant factor, \( s \approx 7.04n_\gamma \), so that

\[
\eta \approx 7B \approx 2.68 \times 10^{-8}(\Omega_B h^2).
\]

As we will discuss in the next Chapter, primordial nucleosynthesis constrains \( \eta \) to the interval \((4\ to\ 7) \times 10^{-10}\), corresponding to \( B \approx (6\ to\ 10)\)
The inverse of $B$, $s/n_B \simeq (1 \text{ to } 2) \times 10^{10}$, is the entropy per baryon; we live in a Universe with very high entropy.

### 3.6 Horizons

In Section 2.2 it was shown that the proper distance to the horizon in a Robertson-Walker space is given by

$$d_H(t) = R(t) \int_0^t \frac{dt'}{R(t')}.$$  \hspace{1cm} (3.105)

If $R(t) \propto t^n$, then for $n < 1$, $d_H(t)$ is finite and equal to $t/(1 - n)$. That is, in spite of the fact that all physical distances approach zero as $R \to 0$, the expansion of the Universe precludes all but a tiny fraction of the Universe from being in causal contact. This is a vexing feature of the standard cosmology to which we will return again. If curvature effects can be neglected (which is a good approximation in the early Universe), the Friedmann equation ($\dot{R}/R = (8\pi G \rho/3)^{1/2}$) implies $R(t) \propto t^{2/(3(1 + w))}$ for $p = \omega \rho$. For a radiation-dominated Universe, $R \propto t^{1/2}$, and $d_H(t) = 2t$, while for a matter-dominated Universe $R \propto t^{2/3}$, and $d_H(t) = 3t$.

It is straightforward to find a more general expression for $d_H$ that includes the effect of curvature. The integral for $d_H$ can be written as

$$d_H(t) = R(t) \int_0^t \frac{dt'}{R(t')} = R(t) \int_0^{R(t)} \frac{dR(t')}{R(t') R(t')}.$$ \hspace{1cm} (3.106)

Generalizing (3.20) for $\rho \propto R^{-3(1 + w)}$,

$$\dot{R}^2 = R_0^2 H_0^2 \left[ 1 - \Omega_0 + \Omega_0 \left( \frac{R_0}{R} \right)^{1 + 3w} \right],$$ \hspace{1cm} (3.107)

the expression for $d_H(t)$ becomes

$$d_H(t) = \frac{1}{H_0(1 + z)} \int_0^{(1+z)^{-1}} \frac{dx}{[x^2(1 - \Omega_0) + \Omega_0 x^{(1 - 3w)}]^{1/2}}.$$ \hspace{1cm} (3.108)

\footnote{Here, and throughout, $\eta$ is defined as the value of the baryon-to-photon ratio at the present epoch.}